

The Figure Eight Knot Group is Conjugacy Separable

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Abstract

We prove that torsion free subgroups of $PGL_2(\mathbb{C})$ (abstractly) commensurable with the Euclidean Bianchi groups are conjugacy separable. As a consequence we deduce the result stated in the title.

1 Introduction

A group G is conjugacy separable if whenever x and y are non-conjugate elements of G , there exists some finite quotient of G in which the images of x and y are non-conjugate. The notion of the conjugacy separability owes its importance to the fact, first pointed out by Mal'cev [M-58], that the conjugacy problem has a positive solution in finitely presented conjugacy separable groups. It is known that free products of conjugacy separable groups are again conjugacy separable (see [R-71]). The property is not preserved in general by the formation of free products with amalgamation and HNN -extensions. The most general result on conjugacy separability of free products with cyclic amalgamation was proved in [R-S-Z-98]. To investigate the property for a non-cyclic amalgamation is much more difficult and only few results exist on the subject (see [R-T-V-95] and [Z-T-95]). One of the most interesting examples of free amalgamated products are Bianchi groups, i.e., the groups $PSL_2(O_d)$, where O_d denotes the ring of integers of the field $\mathbb{Q}(\sqrt{-d})$ for each square-free positive integer d . These groups have long been of interest, not only because of their intrinsic interest as abstract

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groups, but also because they arise naturally in number theory and geometry. In [W-Z-98] conjugacy separability of Bianchi groups $PSL_2(O_d)$ for $d = 1, 2, 7, 11$ was proved and conjectured for the rest of them.

In this article we prove the conjugacy separability for torsion free subgroups of $GL_2(\mathbb{C})$ (abstractly) commensurable with the Euclidean Bianchi groups, i.e., groups $PSL_2(O_d)$, $d = 1, 2, 3, 7, 11$. We observe that it is not known whether a group commensurable with a conjugacy separable group G (i.e., having a subgroup of finite index isomorphic with some subgroup of finite index of G) is conjugacy separable.

As a consequence of our main result we deduce that the figure eight knot group is conjugacy separable. Conjugacy separability for knot groups is not known in general, however is known for the trefoiled knot group since it is a free product of cyclic groups with cyclic amalgamation.

From a group theoretic point of view conjugacy separability of a group indicates that it has many subgroups of finite index. The other property that indicates this is subgroup separability. Recall that a group is called subgroup separable (or LERF) if its finitely generated subgroups are closed in the profinite topology. It follows from the definition that commensurability preserves subgroup separability. Although conjugacy separability and subgroup separability do not imply each other, quite often they hold together. The subgroup separability of Bianchi groups was proved recently (see [L-R-2004, Theorem 3.6.1]). Combining this with the results of the paper we deduce that a torsion-free group commensurable with any finitely generated subgroup of an Euclidean Bianchi group is conjugacy separable.

The methods used in the paper are based on Bass-Serre theory of groups acting on trees and its profinite version. The basic notions of these theories are freely used here and can be found in [S-77] and [Z-M-89], [Z-89], [Z-M-90] respectively.

For a group G and elements $g, h \in G$ we shall frequently use notation $g^h = h^{-1}gh$ in the paper.

2 Preliminary results

2.1 Centralizers

Lemma 2.1. *If $g \in G = GL_2(\mathbb{C}), SL_2(\mathbb{C})$ such that $g \notin Z(G)$, then $C = C_G(g)$ is abelian.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$, $ad - bc \neq 0$. Using the Jordan

canonical form we may assume that $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $ad \neq 0$. To calculate the centralizer of g , we have two cases to consider.

Case 1. The eigenvalues of g are distinct. Hence g is diagonal and so $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Then a straightforward calculation shows that the centralizer of g is equal to

$$C_{GL_2(\mathbb{C})}(g) = \left\{ \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} \mid xw \neq 0 \right\}.$$

Case 2. If $a = d$, i.e., the eigenvalues are equal, a straightforward calculation shows that the centralizer of g is

$$C_{GL_2(\mathbb{C})} = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x \neq 0 \right\}.$$

Thus in both cases the centralizer of a non-central element $g \in GL_2(\mathbb{C})$ is abelian.

Finally observe that $C_{GL_2(\mathbb{C})}(g) \cap SL_2(\mathbb{C}) = C_{SL_2(\mathbb{C})}(g)$ and therefore the centralizer of a non-central element of $g \in SL_2(\mathbb{C})$ is also abelian. \square

Lemma 2.2. *Let $\bar{g} \in PGL_2(\mathbb{C})$ be an element of infinite order. Then the centralizer $C_{PGL_2(\mathbb{C})}(\bar{g})$ of \bar{g} is abelian. Moreover, $C_{\Gamma_d}(\bar{g})$ is virtually free abelian of rank at most 2, where $\Gamma_d = PSL_2(O_d)$.*

Proof. Consider the projection $\varphi : GL_2(\mathbb{C}) \longrightarrow PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/Z$, where $Z = Z(GL_2(\mathbb{C})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C} \right\}$ is the center of $GL_2(\mathbb{C})$. We use \bar{g} for the image of g in $GL_2(\mathbb{C})/Z$. Pick $1 \neq \bar{g} \in PGL_2(\mathbb{C})$.

We write down the inverse image of the centralizer of the element $\bar{g} \in PGL_2(\mathbb{C})$:

$$\varphi^{-1}(C_{PGL_2(\mathbb{C})}(\bar{g})) = \{h \in GL_2(\mathbb{C}) \mid hgh^{-1} = gz, \text{ for some } z \in Z\}.$$

Using the Jordan canonical form we may suppose that $g = \begin{pmatrix} x & y \\ 0 & w \end{pmatrix}$ and

$gz = \begin{pmatrix} ax & ay \\ 0 & aw \end{pmatrix}$. Since conjugate matrices have the same eigenvalues, these matrices can be conjugate in $GL_2(\mathbb{C})$ only if either $a = 1$ or $aw = x$ and $w = ax$. In the second case we have $a^2 = 1$ and so $a = -1$.

But the matrix $g = \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix}$ has finite order in $PGL_2(\mathbb{C})$, because $g^2 = \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix}$ is in the center of $GL_2(\mathbb{C})$. Since g has infinite order by hypothesis, $a = 1$.

Thus we have

$$\varphi^{-1}(C_{PGL_2(\mathbb{C})}(\bar{g})) = C_{GL_2(\mathbb{C})}(g).$$

Since $Z \leq C_{GL_2(\mathbb{C})}(g)$, we have that $C_{PGL_2(\mathbb{C})}(g)$ is the quotient group of the centralizer of g in $GL_2(\mathbb{C})$ modulo the center, i.e., $C_{PGL_2(\mathbb{C})}(g) \cong C_{GL_2(\mathbb{C})}(g)/Z$. By Lemma 2.1 the centralizer of g in $PGL_2(\mathbb{C})$ is abelian.

The second part of the lemma follows from the fact that the torsion free abelian subgroups in Γ_d are free abelian of rank at most 2, see Fine [F-89] in page 107. \square

Corollary 2.3. *A subgroup G of $PGL_2(\mathbb{C})$ does not contain the generalized dihedral group. In particular, this is so for Bianchi groups.*

Proof. If the generalized dihedral group $E \cong \langle x, y | x^2 = y^2 \rangle$ is a subgroup of $PGL_2(\mathbb{C})$ then $E \leq C_G(x^2)$ which is an abelian group by Lemma 2.1. \square

2.1 The Profinite topology

The profinite topology on a group G is the topology where the collection of all finite index normal subgroups of G serves as a fundamental system of neighborhoods of the identity element $1 \in G$, turning G into a topological group. The completion \widehat{G} of G with respect to this topology is called the profinite completion of G and can be expressed as an inverse limit

$$\widehat{G} = \varprojlim_{N \in \mathcal{N}} G/N,$$

where $\mathcal{N} = \{ N \mid N \trianglelefteq_f G \}$. Thus \widehat{G} becomes a profinite group, i.e. a compact totally disconnected topological group. Moreover, there exists a natural homomorphism $\iota : G \longrightarrow \widehat{G}$ that sends $g \mapsto (gN)$, this homomorphism is a monomorphism when G is residually finite. If S is a subset of a topological group \widehat{G} , we denote by \overline{S} its closure in \widehat{G} . The profinite topology on G is induced by the topology of \widehat{G} . Note that for a subgroup H of G , the profinite topology of H can be stronger than the topology induced by the profinite topology of G .

Next proposition expresses the conjugacy separability property of G in terms of the profinite topology and we shall use it freely in the paper.

Proposition 2.4. *Let G be a group, then the following conditions are equivalent:*

- (i) G is conjugacy separable;
- (ii) for each $x \in G$, the conjugacy class x^G of x is closed in the profinite topology. In particular G is residually finite;
- (iii) G is residually finite and for each pair of elements $x, y \in G$ such that $y = x^\gamma$, for some $\gamma \in \hat{G}$, there exists $g \in G$ such that $y = x^g$.

Lemma 2.5. *Let G be a residually finite group, U a finite index subgroup of G and H a subgroup of G . Then $\bar{U} \cap \bar{H} = \overline{U \cap H}$.*

Proof. We just need to proof that $\bar{U} \cap \bar{H} \leq \overline{U \cap H}$, since clearly $\overline{U \cap H} \leq \bar{U} \cap \bar{H}$.

Consider the following commutative diagram

$$\begin{array}{ccc}
 G & \xrightarrow{\quad} & \hat{G} \\
 \uparrow & \searrow & \uparrow \\
 H & \xrightarrow{\quad} & \bar{H} \\
 \searrow \varphi_H & & \swarrow \varphi_{\bar{H}} \\
 & G/U = \hat{G}/\bar{U} &
 \end{array}$$

Note that $\text{Im} \varphi_{\bar{H}} = \text{Im} \varphi_H$, $\text{Ker}(\varphi_{\bar{H}}) = \bar{H} \cap \bar{U}$ and $\text{Ker}(\varphi_H) = H \cap U$. Therefore, $[\bar{H} : (\bar{H} \cap \bar{U})] = |\text{Im} \varphi_{\bar{H}}| = |\text{Im} \varphi_H| = [H : (H \cap U)] \geq [\bar{H} : \overline{H \cap U}]$. It follows that $\bar{U} \cap \bar{H} = \overline{U \cap H}$ as needed. \square

Let $G = \pi_1(\mathcal{G}, \Gamma)$ be a fundamental group of the graph of conjugacy separable groups (\mathcal{G}, Γ) . Then the profinite topology on G might induce a weaker topology on the vertex and edge groups than their own profinite topologies. This prevents to use effectively the conjugacy separability of the vertex and edge groups for proving the conjugacy separability of G . Thus it is natural to assume that the profinite topology on G is reasonably “strong”. We formalize it in the following definition.

Definition 2.6. *Let G be a group acting on a tree S such that S/G is finite. We say that the profinite topology on G is efficient, if G is residually finite, the edge stabilizers G_e and the vertex stabilizers G_v are closed in the profinite topology on G and the profinite topology of G induces the full profinite topology on the vertex and edge stabilizers.*

If S/G is finite, then we can define a profinite graph

$$\widehat{S} = \varprojlim_{U \triangleleft_f G} S/U,$$

where U ranges over the normal subgroups of finite index of G . The profinite completion \widehat{G} acts continuously on \widehat{S} and $\widehat{S}/\widehat{G} = S/G$. Moreover, \widehat{S} is a profinite tree (see Proposition 3.8 in [Z-M-89]).

Remark 2.7. *Suppose the profinite topology on G is efficient. Then S embeds naturally in \widehat{S} . This follows from the fact that G/G_m embeds in $\widehat{G}/\widehat{G}_m$ because G_m are closed in G for all $m \in S$. Moreover, S is dense in \widehat{S} .*

We define the standard trees $S(G)$ on which G acts (respectively, $S(\widehat{G})$ on which \widehat{G} acts) only for the cases of an amalgamated free product $G = G_1 *_H G_2$ (respectively, $\widehat{G} = \widehat{G}_1 \amalg_{\widehat{H}} \widehat{G}_2$) and an HNN-extension $G = \text{HNN}(G_1, H, t)$ (respectively, $\widehat{G} = \text{HNN}(\widehat{G}_1, \widehat{H}, t)$) since we shall use them only for these cases.

- Let $G = G_1 *_H G_2$. Then the vertex set is $V(S(G)) = G/G_1 \cup G/G_2$, the edge set is $E(S(G)) = G/H$, and the initial and terminal vertices of an edge gH are respectively gG_1 and gG_2 .
- Similarly, let $\widehat{G} = \widehat{G}_1 \amalg_{\widehat{H}} \widehat{G}_2$. Then the vertex set is $V(S(\widehat{G})) = \widehat{G}/\widehat{G}_1 \cup \widehat{G}/\widehat{G}_2$, the edge set is $E(S(\widehat{G})) = \widehat{G}/\widehat{H}$, and the initial and terminal vertices of an edge $g\widehat{H}$ are respectively $g\widehat{G}_1$ and $g\widehat{G}_2$.
- Let $G = \text{HNN}(G_1, H, t)$. Then the vertex set is $V(S(G)) = G/G_1$, the edge set is $E(S(G)) = G/H$, and the initial and terminal vertices of an edge gH are respectively gG_1 and gtG_1 .
- Similarly Let $\widehat{G} = \text{HNN}(\widehat{G}_1, \widehat{H}, t)$. Then the vertex set is $V(S(\widehat{G})) = \widehat{G}/\widehat{G}_1$, the edge set is $E(S(\widehat{G})) = \widehat{G}/\widehat{H}$, and the initial and terminal vertices of an edge $g\widehat{H}$ are respectively $g\widehat{G}_1$ and $gt\widehat{G}_1$.

It follows that $S(\widehat{G}) = \widehat{S(\widehat{G})}$ and so Remark 2.7 applies for $S(G)$.

Lemma 2.8. *Let $G = G_1 \amalg G_2 \amalg \cdots \amalg G_k$ be a free profinite product of profinite groups. Then G_i are isolated in G , i.e., if $g^n \in G_i$, for some natural number n and some $g \in G$ of infinite order, then $g \in G_i$.*

Proof. By an obvious induction on the number of free factors it suffices to consider the case $k = 2$. Let $g \in G$ be an element such that $g^n \in G_i$, we show that $g \in G_i$, for some $i = 1, 2$. Consider the action of G on the standard profinite tree $S(G)$ (one can use its construction above for

amalgamated free product and take an amalgamating subgroup to be trivial). Then g^n fixes a vertex in $S(G)$. Hence $\langle g \rangle / \langle g^n \rangle$ is a finite group acting on the subtree of fixed points T^{g^n} (see Theorem 2.8 in [Z-M-89]). By Theorem 2.10 in [Z-M-89] $\langle g \rangle / \langle g^n \rangle$ fixes a vertex in T^{g^n} and so g fixes the same vertex, say hG_j . It follows that $hgh^{-1} \in G_j$. Since $G_i \cap hG_jh^{-1} = 1$ for $i \neq j$ as one can easily see mapping G onto $G_1 \times G_2$ we deduce that $i = j$. Since $G_i \cap G_i^x = 1$ for $x \in G \setminus G_i$ (see Theorem 9.1.12 in [RZ]) $g \in G_i$, as needed. \square

We distinguish two types of elements of a group G acting on a tree S : we say that a non-trivial element g is *hyperbolic* if it does not stabilize any vertex of S . Otherwise, g is called *non-hyperbolic*. The following proposition will be used in some of our proofs.

Proposition 2.9 (Proposition 24 in [S-77]). *Let G be an amalgamated free product or an HNN-extension and $S = S(G)$ its standard tree. Suppose that an element $a \in G$ is hyperbolic. Put $m = \min_{v \in V(S)} l[v, av]$ and $T_a = \{v \in V(S) \mid l[v, av] = m\}$, where $l[v, av]$ represents the length of the geodesic $[v, av]$. Then T_a is the vertex set of a straight line (that is, a doubly infinite chain of $S(G)$), on which a acts as a translation of amplitude m . Moreover, all $\langle a \rangle$ -invariant subtrees of $S(G)$ contain T_a . Finally if $v \in T_a$, then $T_a = \langle a \rangle[v, av[$.*

The following proposition gives information on the closure of a straight line in $S(\widehat{G})$, and shows that a hyperbolic element of G is hyperbolic in the profinite sense.

Proposition 2.10. *Let G be an amalgamated free product or an HNN-extension and $S = S(G)$ its standard tree. Suppose also that the profinite topology on G is efficient. Let $a \in G$ be a hyperbolic element and T_a the corresponding straight line. Then:*

- (i) $\overline{\langle a \rangle}$ acts freely on the tree $S(\widehat{G})$;
- (ii) $\overline{T_a} = \overline{\langle a \rangle}[v, av[$, where v is a vertex of T_a ;
- (iii) $\overline{\langle a \rangle}$ acts freely on the profinite tree $\overline{T_a}$, and $\overline{\langle a \rangle} \cong \widehat{\mathbb{Z}}$;
- (iv) T_a is a connected component of $\overline{T_a}$ considered as an abstract graph, in other words, the only vertices of $\overline{T_a}$ that are at a finite distance from a vertex of T_a are those of T_a .

Proof. Item (i) is proved in Proposition 2.9 [R-Z-96]. Items (ii) and (iii) are proved in Lemma 4.1 [R-S-Z-98].

To prove (iv) we use the same argument as in Lema 4.3 [R-S-Z-98]. Let w be a vertex of $\overline{T_a}$ that is at finite distance from v . We show that $w \in T_a$. Suppose that $w \notin T_a$, then there exists an edge e' of the geodesic $[v, w]$ that is not in T_a and we can assume that the initial vertex of e' is v . Since $\overline{T_a} = \langle a \rangle[v, av[$, there exists $\alpha \in \langle a \rangle$ such that $\alpha e = e'$, where e is an edge of $[v, av[$. Let w_0 be the origin of the edge e , then $\alpha w_0 = v$. Thus $\alpha w_0 = v \in T_a$, and since $T_a = \langle a \rangle[v, av[$ there exists $\beta \in \langle a \rangle$ such that $\beta \alpha w_0 \in [v, av[$, so $\beta \alpha = 1$, i.e. $\alpha \in \langle a \rangle$. Then $\alpha e = e' \in T_a$ and so $w \in T_a$, a contradiction. \square

We shall need the following generalization of the M. Hall Theorem to free products. Remind that a group G is called LERF if each finitely generated subgroup is closed in the profinite topology of G .

Proposition 2.11 ([G-M-S-2003]). *Let G_1, \dots, G_m be LERF groups and let H be a finitely generated subgroup of the free product $G = G_1 * \dots * G_m$. Then there exists a subgroup U of G of finite index and (Kurosh-type) decompositions*

$$U = U_1 * \dots * U_t \text{ and } H = H_1 * \dots * H_t$$

such that

- (a) $H_i \leq U_i$, $i = 1, \dots, t$;
- (b) For each $i = 1, \dots, t-1$, U_i is a subgroup of finite index of a conjugate of some G_j , $j = 1, \dots, m$, i.e., $U_i = \tau G_j \tau^{-1}$ for some $\tau \in G$;
- (c) U_t is a free group of finite rank and H_t is a free factor of U_t .

Sometimes we need to separate the conjugacy class of a subset or a subgroup of G in a finite quotient rather than the conjugacy class of an element. The following definition formalizes this.

Definition 2.12. *A subgroup H of a group G is called conjugacy distinguished, if $\cup_{g \in G} H^g$ is closed in the profinite topology of G . Or equivalently, for every element $g \in G$ such that $g^\gamma \in \overline{H}$, where $\gamma \in \widehat{G}$, there exists $\delta \in G$ such that $g^\delta \in H$.*

The following proposition collects important results about free product of virtually abelian and virtually free groups that will be used in the paper.

Proposition 2.13. *Let G be a finitely generated free product of virtually abelian groups and a virtually free groups, then:*

1. G is conjugacy separable;

2. G is LERF;
3. The profinite topology on G induces the (full) profinite topology on every finitely generated subgroup of G ;
4. For each pair of finitely generated subgroups H_1, H_2 of G , the product $H_1 H_2$ is a closed subset of G , i.e., $\overline{H_1 H_2} \cap G = H_1 H_2$, where $\overline{H_1 H_2}$ is the closure of $H_1 H_2$ in \widehat{G} ;
5. Every finitely generated subgroup H of G is conjugacy distinguished;

Proof. Item 1 follows from the fact that a free product of conjugacy separable groups is conjugacy separable and that virtually abelian and virtually free groups are conjugacy separable (see [Dy-79] and [R-71]). Item 2 follows from the fact that a free product of LERF groups is LERF (see [Ro-69] and [B-71]). Item 3 follows from 2, because every subgroup H of finite index of a finitely generated subgroup K is finitely generated and so is closed in the profinite topology of G . Indeed, this means that H is the intersection of subgroups of finite index of G and so for some of them, say U one has $U \cap K = H$. Item 4 was proved in [R-Z-04] and [C-2001].

5. Let $g \in G$ and suppose that $g^\gamma \in \bar{H}$, for some $\gamma \in \widehat{G}$. Let $\langle g \rangle$ be the subgroup generated by g . Then by Proposition 2.11 there is a subgroup U of finite index such that $\langle g \rangle \leq U \leq G$, and $\langle g \rangle$ is a subgroup of a free factor of the Kurosh decomposition for U . Hence g is a non-hyperbolic element of U . Since $U \leq_f G$, $\widehat{G} = \widehat{U}G$, so $\gamma = u\sigma$, where $u \in \widehat{U}$ and $\sigma \in G$. Therefore substituting H by $H^{\sigma^{-1}}$ we may assume that $\gamma \in \widehat{U}$.

Now observe that $H \cap U$ is a finitely generated subgroup of U , because $H \cap U$ has finite index in H . Hence by Proposition 2.11, there exists a finite index subgroup U_0 of U such that $H \cap U \leq U_0 \leq U$ and the Kurosh decomposition of $H \cap U$ is compatible with the Kurosh decomposition of U_0 . Namely the decompositions of U_0 and $H \cap U$, are respectively as follows

$$\begin{aligned} U_0 &= U_0^1 * \dots * U_0^m \\ H \cap U &= H_1 * \dots * H_m, \end{aligned}$$

where $H_i \leq U_0^i$. Since $U_0 \leq_f U$, $\widehat{U} = \widehat{U_0}U$, so $\gamma = g'\delta$, where $g' \in \widehat{U_0}$ and $\delta \in U$. Thus substituting U_0, H and $H \cap U$ (together with their Kurosh decompositions) by conjugates $U_0^{\delta^{-1}}, H^{\delta^{-1}}$ and $(H \cap U)^{\delta^{-1}}$ in G , we can suppose that $\gamma \in \widehat{U_0}$. Since $g^\gamma \in \bar{H}$ and $g^\gamma \in \bar{U}$, one has $g^\gamma \in \bar{H} \cap \bar{U}$.

Since $U_0 \leq_f U$, then $g^n \in U_0$ for some natural number n . Since g is an element of a free factor of U , g^n is an element of a conjugate of a free factor of U_0 , so without loss of generality we suppose that $g^n \in U_0^1$. Now

$$\bar{U_0} = \widehat{U_0} = \widehat{U_0^1} \amalg \dots \amalg \widehat{U_0^m}.$$

Then by the projective limit argument one deduces that $g^{n\gamma}$ is conjugate to an element of a free factor of

$$\overline{H \cap U} = \bar{H}_1 \amalg \dots \amalg \bar{H}_m.$$

Indeed, $\bar{U}_0 = \varprojlim \bar{U}_{0V}$, where

$$U_{0V}^- = \hat{U}_0^1 / (\hat{U}_0^1 \cap V) \amalg \dots \amalg \hat{U}_0^m / (\hat{U}_0^m \cap V)$$

and V ranges over all open subgroups of \bar{U}_0 . Then

$$\overline{(H \cap U_0)_V} = \hat{H}_0^1 / (\hat{H}_0^1 \cap V) \amalg \dots \amalg \hat{H}_0^m / (\hat{H}_0^m \cap V).$$

Since the image $g_V^{n\gamma}$ of $g^{n\gamma}$ in $\overline{(H \cap U_0)_V}$ has finite order, by Theorem 1 in [H-R-85] it is conjugate in $\overline{(H \cap U_0)_V}$ to an element of a free factor for every V . Therefore $g^{n\gamma}$ is conjugate in $\overline{H \cap U}$ to an element of \bar{H}_i . By Theorem B' in [H-R-85] $\bar{U}_0^1 \cap H_j^u = 1$ for all $u \in \bar{U}_0$, $j > 1$. Hence $g^{n\gamma}$ is conjugate to \bar{H}_1 in $\overline{U_0 \cap H}$.

Thus we may assume that $g^{n\gamma} \in \bar{H}_1$. By Lemma 2.5 $\overline{H \cap U_0} = \overline{H \cap U_0}$, therefore, $g^\gamma \in \bar{H}_1$, since \bar{H}_1 is isolated in $\overline{H \cap U} = \bar{H}_1 \amalg \dots \amalg \bar{H}_m$ (see Lemma 2.8). Since $g^n \in U_0^1$ and $g^{n\gamma} \in U_0^1$, by Theorem 9.1.12 in [R-Z-96] the conjugating element $\gamma \in \bar{U}_0^1$. It follows that $g \in U_0^1 = G \cap \bar{U}_0^1$. But U_0^1 is either virtually free or virtually abelian, so H_1 is conjugacy distinguished in U_0^1 . This finishes the proof. \square

If Γ is a profinite graph then its connected component is a maximal connected profinite subgraph of Γ . Note that it coincide with the closure of a connected component of Γ considered as an abstract graph, since the closure of a connected abstract subgraph is a connected profinite subgraph. If a profinite group G acts on Γ and C its connected component we denote by $Stab_G(C)$ the maximal subgroup of G leaving C invariant.

Lemma 2.14. *Let G be a profinite group acting on a profinite graph S and let m_1, m_2 be elements of a connected component C of S . Then*

$$C / Stab_G(C) \subseteq S / G.$$

Proof. We need to show that if $g \in G$ with $gm_1 = m_2$, then g leaves C invariant, i.e. $g \in Stab_G(C)$. Since $m_2 \in C \cap gC$, $C \cup gC$ is connected and so $C = gC$. \square

Lemma 2.15. *Let G be a group acting on a profinite tree S such that S/G is finite. Let Δ be a profinite subgraph of S and e be an edge in S . Suppose a connected component C of $\Gamma := \Delta \setminus Ge$ contains at least one edge. Then there exist edges $e' \in C$ and $e_0 \in Ge \cap \Delta$ that have a common vertex v .*

Proof. Consider the profinite graph Δ_Γ obtained from Δ by collapsing all connected components of Γ . By Proposition 3.2 in [Z-92] Δ_Γ is a profinite tree. Let v_C be the vertex that is the image of the connected component C . Now since $E(\Delta_\Gamma)$ is compact (because $\Delta \setminus \widehat{Ge}$ is open and closed), by Proposition 2.15 in [Z-M-89] there exists an edge $\overline{e_0} \in \Delta_\Gamma$ that has v_C as a vertex, say $v_C = d_0(\overline{e_0})$. Let $e_0 = \pi_\Gamma^{-1}(\overline{e_0})$, where $\pi_\Gamma : \Delta \rightarrow \Delta_\Gamma$ is a quotient map. Then $d_0(e_0) \in C$ and since $E(C) = E(S(G)) \cap C$ is compact, by Proposition 2.15 [Z-M-89] there exist $e' \in E(C)$ having $d_0(e_0)$ as a vertex. Note also that since $\pi_\Gamma(E(\Delta)) = \pi_\Gamma(\widehat{Ge})$ one has $e_0 \in \widehat{Ge}$. \square

Lemma 2.16. *Let G be a group that acts on a tree S such that S/G is finite and the edge stabilizers are finitely generated. Suppose that the profinite topology on G is efficient and there exists an epimorphism $\tau : G \rightarrow K$ such that the restriction of τ to the vertex stabilizer G_v is an isomorphism for each $v \in V(S)$. Suppose every finitely generated subgroup of K is conjugacy distinguished. Then the edge stabilizers of G are conjugacy distinguished in G .*

Proof. Let $e \in S$ be an edge. We have to show that G_e is conjugacy distinguished in G . Let $g \in G$ and $\gamma \in \widehat{G}$ be such that $g^\gamma \in \widehat{G_e}$, we need to show that there exists $\gamma_1 \in G$ such that $g^{\gamma_1} \in G_e$. Observe that by item (i) of Proposition 2.10 g is a non-hyperbolic element of G , therefore g stabilizes a vertex v in S . Fix a connected transversal T of S/G in S . Without loss of generality we can suppose that v and e belong to T . We use induction on the number of edges of the geodesic $[v, e]$ in T .

Suppose $[v, e]$ has one edge only. Since τ restricted to G_v is an isomorphism to K , $\tau(G_e)$ is conjugacy distinguished in K by Proposition 2.13 ($\tau(G_e)$ is finitely generated). Hence there exists $k \in K$ such that $\tau(g)^k \in \tau(G_e)$ and since $\tau|_{G_v}$ is an isomorphism to K , there exists $\delta \in G_v$ such that $\tau(\delta) = k$. Then $\tau(g)^{\tau(\delta)} \in \tau(G_e)$, and consequently $g^\delta \in G_e$.

Suppose now that $[v, e]$ has more than one edge. Since g fixes v and $\gamma^{-1}e$ by Theorem 2.8 in [Z-M-89] g fixes the geodesic $[v, \gamma^{-1}e]$, i.e., the minimal profinite subtree which contains v and $\gamma^{-1}e$. Denote by C the connected component of $[v, \gamma^{-1}e] \setminus \widehat{Ge}$ containing v . Note that C contains at least one edge since otherwise v would be a vertex of a translation of e , contradicting the assumption that $[v, e]$ has more than one edge. Then by

Lemma 2.15 there are edges $e' \in C$ and $e_0 \in \widehat{G}e \cap [v, \gamma^{-1}e]$ that have a vertex in common, say $d_0(e_0)$.

Let $\nu : S(\widehat{G}) \rightarrow S(\widehat{G})/\widehat{G}$ be the natural epimorphism. Observe that $\nu(e) = \widehat{G}e$ and $\nu(v) \in (S(\widehat{G})/\widehat{G}) \setminus \nu(e)$. Let $\widehat{\Pi}$ be the maximal subgroup of \widehat{G} that leaves invariant the connected component \widehat{C} of $S(\widehat{G})/\widehat{G}$ containing v , i.e. $\widehat{\Pi} = \text{Stab}_{\widehat{G}}(\widehat{C})$. By Lemma 2.14 $\nu|_{\widehat{C}}$ coincides with the factorization of \widehat{C} modulo $\widehat{\Pi}$. Therefore, replacing γ with its multiple $x\gamma$ for some $x \in \widehat{\Pi}$ we may assume that $e' \in T$. But $[v, e']$ has less edges than $[v, e]$ so by the induction hypothesis g is conjugate to an element of $G_{e'}$ in G . Hence we can assume that $g \in G_{e'}$. It follows that $g \in d_0(e_0)$ and replacing v by $d_0(e_0)$ we may assume that v is the initial vertex of e_0 . But changing γ again we may assume that e_0 is in T (we loosing $e \in T$ of course), so by the induction hypothesis again we deduce that g is conjugate in G to an element of G_{e_0} . However, we observe now that T contains unique edge of the orbit $\widehat{G}e$ so our new $e_0 \in T$ is the same as our old e . Thus g is conjugate to an element of G_e in G as required. \square

Theorem 2.17. *Let G be a torsion free group that does not contain subgroups isomorphic to a generalized dihedral. Let H be a finite index subgroup of G such that:*

1. *H is conjugacy separable;*
2. *for each $h \in H$, $C_H(h)$ is a free abelian group of rank at most 2.*

Then G is conjugacy separable.

Proof. Let $g_1, g_2 \in G$ such that $g_2 = g_1^\gamma$, for some $\gamma \in \widehat{G}$. Since H has finite index in G , $g_1^m, g_2^m \in H$, where $m = |G : H|$.

Observe that $\widehat{G} = G\widehat{H}$, so that we can write $\gamma = d\gamma_0$, where $\gamma_0 \in \widehat{H}$ and $d \in G$. Therefore $g_2^m = (g_1^\gamma)^m = (g_1^m)^\gamma = (g_1^m)^{d\gamma_0}$. Now substituting g_1 by g_1^d , we can suppose that $\gamma \in \widehat{H}$. Thus g_1^m and g_2^m are conjugated in \widehat{H} , and since H is conjugacy separable there exists $h \in H$ such that $g_1^m = (g_2^m)^h$. Hence $g_1^m = (g_2^m)^h$, so g_1^m and g_2^m are conjugate in H . Thus we can suppose that $g_1^m = g_2^m$. Let $N = \langle g_2^m \rangle$ and $K = \langle g_1, g_2 \rangle$. Then K centralizes N . Since $C_G(N)$ is a virtually free abelian group of rank at most 2, K is a virtually free abelian group of rank at most 2. If K is virtually cyclic, then since K is torsion free, by Theorem 3.5 in [D-80] K is cyclic and therefore $g_1 = g_2$.

If K is abelian, $g_1 = g_2$ and we are done. Suppose that K is a non-abelian group having two generated free abelian subgroup of finite index. Choose A a normal torsion free abelian subgroup of finite index, where

without lost generality we can assume that g_1^m is a generator of A . Let $\varphi : K \longrightarrow \text{Aut}(A) = GL_2(\mathbb{Z})$ be the homomorphism induced by the action of K on A . Since N is central in K , $\varphi(K)$ is a finite subgroup of the group of the upper triangular matrices with first column $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$. Hence the other element of the diagonal of this matrix must be 1 or -1 . Since the image of K in $GL_2(\mathbb{Z})$ is finite and a matrix of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ has infinite order, any element of $\varphi(K)$ has the form $\begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}$. But $\begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & y-x \\ 0 & 1 \end{pmatrix}$, so we conclude that $\varphi(K)$ is a group of order 2. Hence the centralizer $P = C_K(A)$ of A in K has index 2. Since $A \leq Z(P)$, $[P, P]$ is finite and since P is torsion free, P is abelian. Therefore we can assume that $P = A$. In this case conjugating the image of K in $GL_2(\mathbb{Z})$ if necessary (i.e., choosing another base for A) we can suppose that $\varphi(K)$ is generated either by the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Indeed, conjugating the matrix $\begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}$ by the matrix $\begin{pmatrix} 1 & y \\ 0 & -1 \end{pmatrix}$ we get $\begin{pmatrix} 1 & 2y-x \\ 0 & -1 \end{pmatrix}$, then we get the first matrix in the case of x being even, and the second matrix in the case of y being odd; in the latter case the matrix is conjugate of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the first case K is a generalized dihedral group. Indeed, let z be an element of the inverse image of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $Z(K)$ is cyclic, because otherwise $Z(K)$ is of finite index and since K is torsion free, K would be abelian. Therefore since z centralizes z^2 , z^2 is a power of g_1^m . But this power can be only 1 or -1 , otherwise factoring out z^2 we get that $K/\langle z^2 \rangle$ has the followings generators $g_1^m \langle z^2 \rangle$, $\alpha \langle z^2 \rangle$ and $z \langle z^2 \rangle$, where α is the generator of the group A inverted by the matrix above; hence we have a 3-generated group, unless z^2 is g_1^m or $(g_1^m)^{-1}$. Thus K has only one relation $\alpha^z = \alpha^{-1}$. Hence K is a generalized dihedral group which is discarded by the hypothesis of the theorem.

In the second case, K has torsion. Indeed, let z be the inverse image of the $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then there exist generators x and y of A that are exchanged by the action of z , i.e., $x^z = y$ and $y^z = x$. Since z centralizes z^2 , $z^2 =$

$(xy)^n$, for some $n \in \mathbb{Z}$. But $(zx^{-n})^2 = zx^{-n}zx^{-n} = z^2(yx)^{-n} = 1$, i.e., zx^{-n} has order 2. Therefore K has torsion and since G is torsion free this case is also discarded.

Thus K is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and this implies that $g_1 = g_2$. \square

3 Principal results

Recall that the Bianchi Groups are $PSL_2(O_d)$, where O_d is the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, and

$$O_d = \mathbb{Z} + w\mathbb{Z}, \text{ where } \begin{cases} w = \sqrt{-d}, \text{ se } -d \equiv 1 \pmod{4} \\ w = \frac{1+\sqrt{-d}}{2}, \text{ se } -d \not\equiv 1 \pmod{4}. \end{cases}$$

and d is a positive square-free integer. Let $m \in \mathbb{N}$ and $O_{d,m} = \mathbb{Z} + mw\mathbb{Z}$, where w is given above. It is easy to see that $|O_d : O_{d,m}| = m$, and we can consider the groups $SL_2(O_{d,m})$. Observe that $SL_2(O_{d,m})$ is of finite index in $SL_2(O_d)$.

We shall use the following decompositions of Bianchi groups $\Gamma_d = PSL_2(O_d)$ for $d = 1, 2, 7, 11$ (see [F-89]).

$$PSL_2(O_1) = G_1 *_M G_2,$$

where $G_1 = S_3 *_{{\mathbb{Z}/3\mathbb{Z}}} A_4$, $G_2 = S_3 *_{{\mathbb{Z}/2\mathbb{Z}}} D_2$ and M is the modular group $PSL_2(\mathbb{Z})$.

$$\begin{aligned} PSL_2(O_2) &= HNN(K_2, M, t) \\ PSL_2(O_7) &= HNN(K_7, M, t) \\ PSL_2(O_{11}) &= HNN(K_{11}, M, t), \end{aligned}$$

where $K_2 = (A_4 *_{{\mathbb{Z}/2\mathbb{Z}}} D_2)$, $K_7 = (S_3 *_{{\mathbb{Z}/2\mathbb{Z}}} S_3)$, $K_{11} = (A_4 *_{{\mathbb{Z}/3\mathbb{Z}}} A_4)$ and M is the modular group $PSL_2(\mathbb{Z})$.

The Bianchi group Γ_3 does not decompose in amalgamated free product or HNN -extension. Therefore we are going to use the decomposition found by Scarth of a certain subgroup of finite index in Γ_3 . The decomposition is described in the next

Proposition 3.1 (Scarth, M. R.). *There exists a subgroup $\Gamma_{3,2} := PSL_2(O_{3,2})$ of Γ_3 of finite index which is an HNN -extension of $K_{3,2}$ with the modular group associated, where $K_{3,2} = S_3 *_{{\mathbb{Z}/3\mathbb{Z}}} D(3, 3, 3)$ and $D(3, 3, 3) = \langle x, y | x^3 = y^3 = (xy)^3 = 1 \rangle$.*

Proof. The group $PSL_2(O_{3,2})$ has the following presentation (see [Sc-03])

$$\langle a, t, w | a^2 = (at)^3 = (w^{-1}awa)^3 = [t, w] = 1 \rangle.$$

Let $v = w^{-1}aw$, then using Tietze transformations $((av)^3 = 1$ iff $(va)^3 = 1$ and $(at)^3 = (wvw^{-1}t)^3 = (wvtyw^{-1})^3 = 1$ iff $(tv)^3 = 1$) we get the following presentation

$$\langle a, t, v, w | a^2 = (at)^3 = (av)^3 = v^2 = (tv)^3, t = w^{-1}tw, v = w^{-1}aw \rangle.$$

Let $K_{3,2} = \langle a, t, v | a^2 = (at)^3 = (av)^3 = v^2 = (tv)^3 \rangle$. We show that $PSL_2(O_{3,2})$ is an HNN-extension of $K_{3,2}$ with $\langle a, t \rangle \cong M \cong \langle v, t \rangle$ associated.

Observe first that $\langle a, t \rangle \cong M$, since have the same presentation. Then $PSL_2(O_{3,2}) = HNN(K_{3,2}, M, w)$.

Let $s = at, m = av$. We get

$$K_{3,2} = \langle a, s, m | a^2 = s^3 = m^3 = (am)^2 = (sm^2)^3 = 1 \rangle,$$

where we used again that $sm^2 = atva$ is of order 3 iff tv is of order 3. Observe that $\langle a, m | a^2 = m^3 = (am)^2 = 1 \rangle \cong S_3$ and a group with presentation $\langle s, \bar{m} | s^3 = \bar{m}^3 = (s\bar{m}^{-1})^3 \rangle \cong D(3, 3, 3)$. Then we have $K_{3,2} = S_3 *_{m=\bar{m}} D(3, 3, 3)$. \square

Lemma 3.2. *There exists a homomorphism $\tau : \Gamma_{3,2} \rightarrow P$, where P is virtually a free product of torsion free abelian groups and a free group such that $\tau|_{K_{3,2}}$ is injective.*

Proof. We use the notation of the preceding proposition. Let $\psi_1 : S_3 \rightarrow S_3$ be the map that sends $a \mapsto am, m \mapsto m^{-1}$. Observe that $\{\psi_1(a), \psi_1(m)\}$ satisfies the relations of the group S_3 , so ψ_1 is an epimorphism by Von Dick's theorem. Then ψ_1 is an automorphism of order 2 because am and m^{-1} generate S_3 . Let $\psi_2 : D(3, 3, 3) \rightarrow D(3, 3, 3)$ be the map that sends $s \mapsto s^{-1}, \bar{m} \mapsto \bar{m}^{-1}$, again we have that ψ_2 is an automorphism of order 2. Note that these two automorphisms agree on the amalgamated subgroup $\langle m \rangle = \langle \bar{m} \rangle$, i.e., $\psi_1(m) = m^{-1} = \bar{m}^{-1} = \psi_2(\bar{m})$, therefore by the universal property of an amalgamated free product there exists a unique homomorphism $\psi : S_3 *_{\mathbb{Z}/3\mathbb{Z}} D(3, 3, 3) \rightarrow S_3 *_{\mathbb{Z}/2\mathbb{Z}} D(3, 3, 3)$ extending ψ_1 and ψ_2 . Since ψ_1 and ψ_2 are automorphisms of order 2 it follows that ψ is an automorphism of order 2 as well.

Consider the group $P = (S_3 *_{\mathbb{Z}/3\mathbb{Z}} D(3, 3, 3)) \rtimes \langle \psi \rangle$. Then the map φ that sends $K_{3,2} \mapsto K_{3,2}$ identically and $w \mapsto \psi$, extends to an epimorphism $\tau :$

$\Gamma_{3,2} \rightarrow P$ by Von Dycks theorem because $\{\varphi(a), \varphi(m), \varphi(\overline{m}), \varphi(s), \varphi(w)\}$ satisfies the relations of $\Gamma_{3,2}$. In addition, $\tau|_{K_{3,2}}$ is a monomorphism to P . Now it remains to note that by the Kurosh subgroup theorem (see [S-77] for the version with amalgamation) any torsion free subgroup of finite index of P is a free product of virtually abelian group and a free group, since $D(3, 3, 3)$ is virtually free abelian. \square

Proposition 3.3. *Let H be finitely generated subgroup of K_{32} and M be the modular group. Then $\overline{H \cap M} = \overline{H} \cap \overline{M}$.*

Proof. We must prove that $\overline{H \cap M} \geq \overline{H} \cap \overline{M}$, since, clearly $\overline{H \cap M} \leq \overline{H} \cap \overline{M}$.

Consider $L = \langle K_{32}, K_{32}^{w^{-1}} \rangle = K_{32} *_M K_{32}^{w^{-1}}$ be a amalgamated free product of K_{32} , where M is a modular group and set $P = \langle H, K_{32}^{w^{-1}} \rangle$. By the subgroup theorem for amalgamated free products we have $P = H *_M K_{32}^{w^{-1}}$. Since Γ_{32} is LERF, all the subgroups H, M, P are closed in the profinite topology of Γ_{32} and this topology induces an efficient profinite topology on P and on L .

It follows that $\widehat{L} = \widehat{K_{32}} \amalg_{\widehat{M}} \widehat{K_{32}^{w^{-1}}}$ and $\widehat{P} = \widehat{H} \amalg_{\widehat{H \cap M}} \widehat{K_{32}^{w^{-1}}}$.

By Exercise 9.2.7 (2) in [RZ] $\widehat{H} \amalg_{\widehat{H \cap M}} \widehat{K_{32}^{w^{-1}}}$ is proper. Obviously, $\widehat{H} \cap \widehat{K_{32}^{w^{-1}}}$ contains $\widehat{H} \cap \widehat{M}$. We show that $\widehat{H} \cap \widehat{K_{32}^{w^{-1}}} = \widehat{H \cap M}$. Indeed, suppose not. Pick an element $h \in \widehat{H} \cap \widehat{K_{32}^{w^{-1}}} \setminus \widehat{H \cap M}$. Then there exist an open normal subgroup U of \widehat{P} such that $hU \notin \widehat{H \cap MU}/U$. Put $N = U \cap H$, $N' = U \cap K_{32}^{w^{-1}}$ and $V = \widehat{H \cap M} \cap U$. Consider the natural homomorphism $f : P \rightarrow H/N \amalg_{\widehat{H \cap M}/V} K_{32}^{w^{-1}}/N'$. Then $f(h) \notin \widehat{H \cap M}/V$. But $f(h) \in f(\widehat{H} \cap \widehat{K_{32}^{w^{-1}}}) \subset (H/N) \cap (K_{32}^{w^{-1}}/N') = \widehat{H \cap M}/V$. So $h \in \text{Ker}(f) \cap \widehat{H \cap M} \leq \widehat{H \cap MU}$ contradicting with the fact that $hU \notin \widehat{H \cap MU}/U$. Thus $\widehat{H} \cap \widehat{K_{32}^{w^{-1}}} = \widehat{H \cap M}$. \square

Proposition 3.4. *Let H be a finite index torsion free subgroup of $G = \Gamma_d$, $d = 1, 2, 7, 11$ or $G = \Gamma_{3,2}$. Then H is conjugacy separable.*

Proof. Let $S(G)$ and $S(\widehat{G})$ be the trees associated with decompositions of G and \widehat{G} . By [L-R-2004, Theorem 3.6.1] G is subgroup separable, therefore the profinite topology on G is efficient, so $S(G)$ is embedded in $S(\widehat{G})$ (see Remark 2.7).

By Lemma 2.2 the centralizers of non-trivial elements of H are free abelian groups of rank at most 2. Hence by Theorem 2.17 it is enough to proof the proposition for an appropriate subgroup of finite index of H . An

appropriate subgroup for us is a subgroup that satisfies the conditions of Lemma 2.16. We show the existence of such a subgroup. Indeed, by Lemmas 4.2 and 4.3 in [W-Z-98] and by Lemma 3.2 in the case $d = 3$, there exists virtually a free product P of torsion free abelian groups and a free group and an epimorphism $\tau : G \rightarrow P$ such that $\tau|_{G_v}$ is injective, for all $v \in V(S)$ and $|P : \tau(G_v)| < \infty$. Therefore replacing H by a subgroup of finite index if necessary we may assume that the image of H is a free product of torsion free abelian groups and a free group. As $S(G)/H$ is finite, there exists a finite number of orbits Hv , $v \in S(G)$, i.e. there exists a finite number of vertex stabilizers in H up to conjugation. Since the index $[P : \tau(G_v)]$ is finite, so is $[P : \tau(H_v)]$. It follows that $\tau(H_v)$ has a finite number of conjugates in P . Therefore $K := \bigcap_{v \in V(S)} \tau(H_v)$ has finite index in P and $L = \tau^{-1}(K) \cap H$ is the desired subgroup. Indeed, $\tau|_{L_v}$ is an isomorphism to K , because $\tau(L_v) = \tau(L \cap H_v) = \tau(L)$, where the last equality follows from the definition of L (to see that the right handside is contained in the left handside let $l \in L$; then $\tau(l) \in K \leq \tau(H_v)$; hence there exist $h_v \in H_v$, $x \in \text{Ker}(\tau)$ such that $l = xh_v$; but $\text{Ker}(\tau) \leq \tau^{-1}(K)$, so $h_v \in L$). Moreover, the vertex and edges stabilizers in L are finitely generated and since G is subgroup separable [L-R-2004, Theorem 3.6.1], the profinite topology on G induces the efficient profinite topology on L . Note also that every finitely generated subgroup of K is conjugacy distinguished by Proposition 2.13, because K is isomorphic to a subgroup of P and so by the Kurosh subgroup theorem satisfy the hypothesis of Proposition 2.13. Thus replacing H by L if necessary, from now on we may assume that H satisfies the hypotheses of Lemma 2.16.

Let $g_1, g_2 \in H$ such that $g_2 = \gamma^{-1}g_1\gamma$ for any $\gamma \in \widehat{H}$. We show that g_1 and g_2 are conjugate in H . We split the proof into two cases.

Case 1: g_1 is non-hyperbolic. By item (i) of Proposition 2.9 in [R-Z-96] g_2 is non-hyperbolic as well. Let v_1 be a vertex fixed by g_1 . If $\gamma \in \widehat{G}_{v_1}$, then $g_2 = g_1^\gamma \in \widehat{G}_{v_1}$. Since $\widehat{G}_{v_1} \cap H = G_{v_1} \cap H$, the elements $g_1, g_2 \in G_{v_1} \cap H$. By Lemma 2.5 $\widehat{G}_{v_1} \cap \widehat{H} = \widehat{G_{v_1} \cap H}$, so $\gamma \in \widehat{G_{v_1} \cap H}$. Since H satisfies the hypothesis of Lemma 2.16, the group $H_{v_1} = G_{v_1} \cap H$ satisfies hypothesis of Proposition 2.13 and so it is conjugacy separable. Thus there exists $\gamma_1 \in G_{v_1} \cap H$ such that $g_2 = g_1^{\gamma_1}$, i.e. g_1 and g_2 are conjugate in $H \cap G_{v_1}$.

Suppose now that $\gamma \notin \widehat{G}_{v_1}$. We shall prove then that some conjugates of g_1 and g_2 in H both stabilize some edge $e \in S$.

By Theorem 3.12 [Z-M-89] $g_1 = \gamma g_2 \gamma^{-1} \in \widehat{G}_{v_1} \cap \gamma \widehat{G}_v \gamma^{-1} \leq \widehat{G}_{\bar{e}}$, where $\bar{e} \in S(\widehat{G})$, v is a vertex of $S(G)$ stabilized by g_2 and $v = d_0(\bar{e})$ or $v = d_1(\bar{e})$. Now $|G_v : H_v| < \infty$, so $\widehat{G}_v / \widehat{H}_v = G_v / H_v$ and we

get $Star_{S(\widehat{G})}(v)/\widehat{H}_v = Star_{S(G)}(v)/H_v$. This implies the existence of an element $\widehat{h} \in \widehat{H}_v$ such that $e_1 := \widehat{h}\bar{e} \in S(G)$.

Then $g_1^{\widehat{h}^{-1}} \in \widehat{H}_{e_1}$ and since $\widehat{h} \in \widehat{H}_v$, and H_{e_1} is conjugacy distinguished in H_v (by item 5 of Proposition 2.13), there exists $h \in H_v$ such that $g_1^h \in H_{e_1}$.

Thus we may assume that $g_1 \in H_e$, where e is an edge of $S(G)$. By Lemma 2.16 the edge stabilizers H_e are conjugacy distinguished in H . Therefore conjugating g_2 by an element of H if necessary we may assume that $g_2 \in H_e$.

Recall now that $\tau : H \longrightarrow K$ is a homomorphism satisfying hypothesis of Lemma 2.16. As g_1, g_2 are conjugate in \widehat{H} , the elements $\tau(g_1)$ and $\tau(g_2)$ are conjugate in \widehat{K} , and since K is conjugacy separable there exists $k \in K$ such that $\tau(g_2) = \tau(g_1)^k$. Then there exists $z \in H \cap G_{v_1}$ such that $\tau(z) = k$, so $\tau(g_2) = \tau(g_1^z)$, and since $g_2, g_1^z \in G_{v_1} \cap H$ and $\tau|_{G_{v_1}}$ is injective, one has $g_2 = g_1^z$.

Case 2: The elements g_1, g_2 are hyperbolic. Let T_{g_1} and T_{g_2} be the infinite straight lines on which g_1 and g_2 act. Let e be any edge of T_{g_1} . Since $g_2 = \gamma^{-1}g_1\gamma$, the group $\langle g_2 \rangle$ acts freely over $\gamma^{-1}\overline{T_{g_1}}$, and by Proposition 2.2(ii) in [R-Z-96] $\gamma^{-1}\overline{T_{g_1}} = \overline{T_{g_2}}$. It follows that $\gamma^{-1}e \in \overline{T_{g_2}} = \overline{\langle g_2 \rangle T_2}$, where $T_2 = [v_2, g_2v_2]$ for any $v_2 \in T_{g_2}$. Let $x \in \overline{\langle g_2 \rangle}$ such that $x\gamma^{-1}e \in T_2$ and $\widehat{\mu} : S(\widehat{G}) \rightarrow S(\widehat{G})/\widehat{H}$ be the natural map. Then $\widehat{\mu}(x\gamma^{-1}e) = \widehat{\mu}(e)$, and since $S(\widehat{G})/\widehat{H} = S(G)/H$ (as one easily checks using the closedness of HG_e in G) there exists $g \in H$ such that $x\gamma^{-1}e = ge \in S$. Therefore $g^{-1}x\gamma^{-1}e = e$ and so $\delta := g^{-1}x\gamma^{-1} \in \overline{H_e}$. Note that $g_1 = \gamma g_2 \gamma^{-1} = \gamma x^{-1}g_2 x \gamma^{-1} = \delta^{-1}g^{-1}g_2 g \delta$. Then substituting g_2 with $g^{-1}g_2 g$ and γ with δ we can suppose that $\gamma \in \overline{H_e}$ and so T_{g_1} and T_{g_2} have a common edge e .

Let P be a geodesic of finite length in T_{g_1} that has e as one of its edges and such that $\gamma \in \overline{I}$, where $I = \bigcap_{e \in E(P)} H_e$. We show that we can assume that

γ can be substituted by an element that belongs to the closure of intersection of the edge stabilizers of a geodesic which strictly contains P . Let $e_1 \in T_{g_1} \setminus P$ be an edge connected to P , and let v be the common vertex of e_1 and P . Let $P^+ = P \cup \{e_1\}$, and put $e_2 = \gamma^{-1}e_1$ so that $e_2 \in \overline{T_{g_2}}$, because $\gamma^{-1}\overline{T_{g_1}} = \overline{T_{g_2}}$. In fact, $e_2 \in T_{g_2}$. Indeed, $\gamma^{-1}[e, e_1] = [e, e_2]$ is finite and contained in $\overline{T_{g_2}}$. Now observe that T_{g_2} is a connected component of $\overline{T_{g_2}}$ considered as an abstract graph, in others words the only vertices of $\overline{T_{g_2}}$ that are at a finite distance from T_{g_2} are the vertices of T_{g_2} , see Proposition 2.10 (iv). Therefore $e_2 \in T_2$.

Let $\mu : S(G) \longrightarrow S(G)/H$ be the natural map. Since $S(G)/H =$

$S(\widehat{G})/\widehat{H}$ we have $\mu(e_1) = \widehat{\mu}(e_1) = \widehat{\mu}(e_2) = \mu(e_2)$, so there exists $h \in H$ such that $e_1 = he_2$. Since e_1 and e_2 have a common vertex v , we have $h \in H_v$. Therefore, $e_2 = h^{-1}e_1 = h^{-1}\gamma e_2$ and so $\gamma_1 := h^{-1}\gamma \in \overline{H_{e_2}}$.

Recall that stabilizers of vertices in H are finitely generated and so by Theorem 1 in [B-C-K-98], I is finitely generated. Since G_v is subgroup separable (Proposition 2.13), the groups I and H_{e_2} are closed in the profinite topology of G_v . Therefore by the main result in [C-2001] IH_{e_2} is closed in the profinite topology of G_v , i.e., $\overline{IH_{e_2}} \cap G_v = IH_{e_2}$. Thus $h = \gamma\gamma_1^{-1} \in IH_{e_2}$ and so there exist $h_1 \in I$, $h_2 \in H_{e_2}$ such that $h = h_1h_2$. Let $\gamma^+ = \gamma h_1^{-1}$, then $\gamma^+e_1 = \gamma h_1^{-1}e_1 = \gamma h_2 h_1^{-1}e_1 = \gamma e_2 = e_1$, and therefore $\gamma^+ \in \overline{H_{e_1}}$. We also have $\gamma^+ = \gamma h_1^{-1} \in \overline{I}$, and since I and H_{e_1} are finitely generated in H_v , by Proposition 2.4 in [W-Z-98] for the cases $d = 1, 2, 7, 11$ and by Lemma 3.3 for the case $d = 3$ we have $\gamma^+ \in \bigcap_{e \in P^+} \overline{H_e}$. Therefore replacing γ with γ^+ and g_2 with $h_1g_2h_1^{-1}$ we may assume that $\gamma \in \bigcap_{e \in P^+} \overline{H_e}$.

Thus we always can assume that P has sufficiently large length. Let P be a geodesic of T_{g_1} that contains e and g_1e . Let $I = \bigcap_{e \in P} H_e$ and consider

$D = I \cap g_1Ig_1^{-1}$. Note that $g_1Ig_1^{-1}$ is the intersection of the stabilizers of the path g_1P . Now by what was done above, we can suppose that $\gamma \in \overline{D}$ (we use $P^+ = P \cup g_1P$ and if $e' = g_1e$, then $H_{e'} = g_1H_eg_1^{-1}$). Observe that I is cyclic, because the intersection of two edge groups in $S(G)$ is cyclic, by Lemma 4.1 in [W-Z-98]. Now we prove that g_1 normalizes D . Indeed, since D is closed in G , then $D = \bigcap_{N \trianglelefteq_f G} DN$. Thus it is enough to prove that DN is normalized by g_1 , for each $N \trianglelefteq_f G$. Indeed, let $N \trianglelefteq_f G$ and consider the quotient G/N . Then the group DN/N have the same index m in the groups IN/N and $I^{g_1^{-1}}N/N$, because these groups are conjugate. If xN generates IN/N , then $(xN)^m$ and $(g_1N)(xN)^m(g_1N)^{-1}$ generate DN/N and we conclude that DN is normalized by g_1 .

Let d be a generator of D and write $E = \langle d, g_1 \rangle$. We prove that E is abelian. Indeed, if d does not centralize g_1 , $g_1dg_1^{-1} = d^{-1}$ and so E is a generalized dihedral group. However, by Corollary 2.3 the Bianchi groups do not contain subgroups isomorphic to the generalized dihedral group, a contradiction.

Therefore d centralizes g_1 and consequently E is abelian. This implies that \overline{E} is abelian and so $g_1 = g_2$, as needed. \square

Theorem 3.5. *Let G be a torsion free group commensurable with a Euclidean Bianchi group. Suppose that G does not contain subgroups isomorphic to the generalized dihedral group. Then G is conjugacy separable.*

Proof. Since G is commensurable with Γ_d (note that commensurability with Γ_3 is equivalent to commensurability with $\Gamma_{3,2}$), there exist isomorphic subgroups M, N of finite index in G and Γ_d respectively. Then by the preceding proposition N is conjugacy separable. Consequently M is conjugacy separable. Then $M \leq G$ satisfies all hypothesis of Theorem 2.17, hence G is conjugacy separable. \square

Theorem 3.6. *Let $G \leq PGL_2(\mathbb{C})$ be a torsion free group commensurable with Γ_d , $d = 1, 2, 3, 7, 11$. Then G is conjugacy separable.*

Proof. Follows from Theorem 3.5 and Corollary 2.3. \square

Corollary 3.7. *The figure eight knot group is conjugacy separable.*

Proof. The figure eight knot group is a torsion free subgroup of index 12 in $\Gamma_3 = PSL_2(O_3)$ (see page 60 in [M-R-02]), so by Theorem 3.6 K is conjugacy separable. \square

Remark 3.8. *In Theorem 3.6 $PGL_2(\mathbb{C})$ can be replaced by $GL_2(\mathbb{C})$ because a subgroup of $GL_2(\mathbb{C})$ containing the generalized dihedral group $\langle x, y \mid x^2 = y^2 \rangle$ must have $x^2 \in Z(GL_2(\mathbb{C}))$ by Lemma 2.1 and so can not be commensurable with Γ_d . Indeed, every subgroup of finite index in Γ_d has finite center.*

References

- [B-71] Burns, R. G., *On Finitely Generated Subgroups of Free Products*, J. Austral. Math. Soc., **12** (1971) 358-364.
- [B-C-K-98] Burns R. G., Chau E. C., Kam S.-M., *On The Rank of Intersection of Subgroups of a Free Product of Groups*, J. Pure Appl. Algebra, **124** (1998), 31-45.
- [C-2001] Coulbois T., *Free Product, Profinite Topology and Partial Action of Groups on Relational Structures: A Connection Between Model Theory and Profinite Topology*, World Scientific, Singapore 2002, 349-361.
- [D-80] Dicks W., *Groups, Trees and Projective Modules*, Lecture Notes in Mathematics, No. 790, Springer-Verlag, 1980.
- [D-D-89] Dicks W., Dunwoody M. J., *Groups Acting on Graphs*, Cambridge Univ. Press., Cambridge, 1989.

- [Dy-79] Dyer J. L., *Separating Conjugates in Amalgamated Free by Finite Groups*, J. London Math. Soc., 2, **20** (1979) 215-222.
- [F-89] Fine B., *Algebraic Theory of the Bianchi Groups*, Marcel Dekker, 1989.
- [G-M-S-2003] Gitik R., Margolis S.W., Steinberg B., *On the Kurosh theorem and separability properties*, Journal of Pure and Applied Algebra, **179** (2003) 87-97.
- [H-R-85] Herfort W. H., Ribes L., *Torsion Elements and Centralizers in Free Products of Profinite Groups*, J. Reine Angew. Math., **358** (1985) 155-161.
- [L-R-2004] Long D. D., Reid A. W., *Subgroup Separability and Virtual Retractions of Groups*, Preprint, 23pp.
- [M-58] Mal'cev A. I., *On Homomorphisms onto Finite Groups*, Uchen. Zap. Ivanovskogo Gos. Ped. Inst., **18** (1958) 40-60.
- [M-R-02] Maclachlan C., Reid A. W., *The Arithmetic of Hyperbolic 3-Manifolds*, Graduate Texts in Mathematics, **219**, Springer, 2002.
- [N-92] Niblo G. A., *Separability Properties of Free Groups and Surface Groups*, J. Pure Appl. Algebra, **78** (1992) 77-84.
- [R-71] Remeslennikov V. N., *Groups that are Residually Finite with Respect to Conjugacy*, Siberian Math. J., **12** (1971) 783-792.
- [Ro-69] Romanovskii V. N., *On the Residual Finiteness of Free Products with Respect to Subgroups*, Izv. Akad. Nauk SSSR Ser. Mat., **33** (1969) 1324-1329.
- [R-T-V-95] Raptis E., Talelli O., Varsos D., *On the Conjugacy Separability of Certain Graphs of Groups*, J. Algebra, **199** (1998) 327-336.
- [R-S-Z-98] Ribes L., Segal D., Zalesskii P. A., *Conjugacy Separability And Free Products of Groups with Cyclic Amalgamation*, J. London Math. Soc., 2, **57** (1998) 609-628.
- [RZ] L. Ribes, P. Zalesskii, *Profinite groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge **40**, Springer-Verlag, Berlin, (2000).
- [R-Z-96] Ribes L., Zalesskii P. A., *Conjugacy Separability of Amalgamated Free Products of Groups*, Journal of Algebra, **179** (1996) 751-774.
- [R-Z-93] Ribes L., Zalesskii P. A., *On the Profinite Topology of a Free Group*, Bull. London Math. Soc., **25** (1993) 37-43.

- [R-Z-04] Ribes L., Zalesskii P.A., *Profinite topologies in Free Products of Groups*. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes. Internat. J. Algebra Comput. 14 (2004), no. 5-6, 751–772.
- [S-70] Serre J.P., *Le Probleme de Groupes de Congruence pour SL_2* , Ann. of Math., **92** (1970) 489-657.
- [S-77] Serre J.P., *Trees*, Springer Monographs in Mathematics, Springer, 1977.
- [Sc-03] Scarth M. R., *PhD Thesis: Normal Congruence Subgroups of Bianchi Groups and Related Groups*, Faculty of Science, University of Glasgow (2003).
- [W-Z-98] Wilson J. S., Zalesskii P. A., *Conjugacy Separability of Certain Bianchi Groups and HNN Extensions*, Math. Proc. Camb. Phil. Soc., **123** (1998) 227-242.
- [Z-M-89] Zalesskii P. A., Mel'nikov O. V., *Subgroups of Profinite Groups Acting on Trees*, Math. Sb., **63** (1989) 405-424.
- [Z-M-90] Zalesskii P. A., Mel'nikov O. V., *Fundamental Groups of Graphs of Profinite Groups*, Algebra i Analiz, **1** (1989) 117-134.
- [Z-T-95] Zalesskii P. A., Tavgen O. I., *Closed Orbits and Finite Approximability With Respect to Conjugacy of Free Almagamated Products*, Mathematical Notes, **58** (1995) 1042-1048.
- [Z-92] Zalesskii P. A., *Open Subgroups of Free Profinite Products*, Contemporary Mathematics, **131** (1992) 473-491.
- [Z-89] Zalesskii P. A., *A Geometrical Characterization of Free Constructions of Profinite Groups*, Siber. Math. J., **30** (1989) 73-84.

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